ISRAEL JOURNAL OF MATHEMATICS **91** (1995), 173–187

SOME RESULTS RELATED TO INTERPOLATION ON HARDY SPACES OF REGULAR MARTINGALES

BY

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ABSTRACT

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n\}_{n\geq 0}$ a regular increasing sequence of sub-σ-fields of $\mathcal{F}$. Let $H_1(\Omega)$ be the usual Hardy space of $\mathcal{F}_n$-martingales. We show that the couple $(H_1(\Omega), L_\infty(\Omega))$ is a partial retract of $(L_1(\Omega), L_\infty(\Omega))$. It is also proved that $(L_p(\Omega), BMO(\Omega))$ is a partial retract of $(L_p(\Omega), L_\infty(\Omega))$ for all $1 < p < \infty$.

1. Introduction

Let $\{\mathcal{F}_n\}_{n\geq 0}$ be an increasing sequence of σ-fields on a probability space $(\Omega, \mathcal{F}, P)$ with $\bigvee_{n\geq 0} \mathcal{F}_n = \mathcal{F}$. In this paper $\{\mathcal{F}_n\}_{n\geq 0}$ will be always assumed **regular** in the sense of [9], that is, each $\mathcal{F}_n$ is generated by a finite partition $\{Q_{n,j}\}_j$ of Ω and there exists a constant M such that

$$(1) \qquad P(Q_{n,j}) \leq M\, P(Q_{n+1,i}), \quad \text{whenever } Q_{n+1,i} \subset Q_{n,j}.$$

The letter M will always denote the constant in the above inequality. A typical example of such sequences of σ-fields is obtained by the dyadic partitions of the unit cube in an Euclidean space. Our general reference for martingale theory is [9].

Let $1 \leq p \leq \infty$. We denote $L_p(\Omega, \mathcal{F}, P)$ simply by L_p and its norm by $\|\cdot\|_p$. Then for a random variable $f \in L_1$ we set, as usual

$$f_n = E(f|\mathcal{F}_n), \quad f^* = \sup_{n\geq 0} |f_n|.$$

Received August 1, 1993 and in revised form February 10, 1994

The Hardy space H_p of martingales on $(\Omega, \mathcal{F}, P)$ relative to $\{\mathcal{F}_n\}_{n\geq 0}$ is defined by

$$H_p = \{f \in L_1: f^* \in L_p\} \quad \text{with } \|f\|_{H_p} = \|f^*\|_p.$$

It is well-known that for $1 < p \leq \infty$, $H_p = L_p$; more precisely, there exists a constant C_p depending only on p such that

$$\|f^*\|_p \leq C_p \|f\|_p, \quad \forall\, f \in L_p, \quad 1 < p \leq \infty. \tag{2}$$

It is also well-known that H_1 can be characterized by the square function:

$$f \in H_1 \iff \Big(|f_0|^2 + \sum_{n\geq 1} |f_n - f_{n-1}|^2\Big)^{1/2} \in L_1.$$

One of the two main results of this paper is the following retractive type theorem concerning the couple (H_1, L_∞).

THEOREM 1: *Given any $f \in H_1$ there exist two linear operators S and T defined on L_1 satisfying the following properties:*

(i) *$S(f) = f^*$ and $|S(g)| \leq g^*$ for any $g \in L_1$; consequently, S is a contraction from H_1 into L_1 and also from L_∞ into L_∞.*

(ii) *$T(f^*) = f$ and T is simultaneously bounded from L_1 into H_1 and from L_∞ into L_∞ with*

$$\|T: L_1 \to H_1\| \leq CM^2, \qquad \|T: L_\infty \to L_\infty\| \leq CM^2,$$

where C is an absolute constant and where M is the constant in (1).

Another main result is about the couple (L_1, BMO). Recall that BMO of martingales on $(\Omega, \mathcal{F}, P)$ relative to $\{\mathcal{F}_n\}_{n\geq 0}$ is the space

$$\mathrm{BMO} = \{f \in L_1: \|\, |f_0| + \sup_{n\geq 1} E(|f - f_{n-1}||\mathcal{F}_n)\|_\infty < \infty\}.$$

In our case where $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, this space can be also defined by

$$\mathrm{BMO} = \{f \in L_1: \|\, |f_0| + \sup_{n\geq 0} E(|f - f_n||\mathcal{F}_n)\|_\infty < \infty\}.$$

It is normed by $\|f\|_{\mathrm{BMO}} = \|f^\sharp\|_\infty$, where $f^\sharp$ is the so-called sharp function of Fefferman–Stein:

$$f^\sharp = |f_0| + \sup_{n\geq 0} E(|f - f_n||\mathcal{F}_n).$$

Recall the well-known inequalities

$$c_p \|f\|_p \le \|f^\sharp\|_p \le C_p \|f\|_p, \quad \forall\, f \in L_p, \quad 1 < p < \infty, \tag{3}$$

where c_p and C_p are two constants depending only on p. For $p = 1$, (3) admits a weak-(1, 1) type inequality. Let $L_{1,\infty}$ be the weak-L_1 space on $(\Omega, \mathcal{F}, P)$. Then

$$c_1 \|f\|_1 \le \|f^\sharp\|_{L_{1,\infty}} \le C_1 \|f\|_1, \quad \forall\, f \in L_1. \tag{4}$$

(3) and (4) are the martingale version of the classical Fefferman–Stein theorem about the sharp function on $\mathbb{R}^n$. They can be proved in the same way as for $\mathbb{R}^n$ (cf., e.g., [8]).

Now we can state our theorem about (L_1, BMO).

THEOREM 2: *Given any $f \in L_1$ there exist two linear operators S and T defined on L_1 satisfying the following properties*

(i) *$S(f) = f^\sharp$ and $|S(g)| \le g^\sharp$ for any $g \in L_1$; consequently, S is simultaneously bounded from L_1 into $L_{1,\infty}$ and from* BMO *into L_∞ with*

$$\|S\colon L_1 \to L_{1,\infty}\| \le C_1, \qquad \|S\colon \mathrm{BMO} \to L_\infty\| \le 1,$$

where C_1 is the constant in (4);

(ii) *$T(f^\sharp) = f$ and T is simultaneously bounded from L_1 into L_1 and from L_∞ into* BMO *with*

$$\|T\colon L_1 \to L_1\| \le CM^2, \qquad \|T\colon L_\infty \to \mathrm{BMO}\,\| \le CM^2,$$

where C is an absolute constant and where M is the constant in (1).

Theorems 1 and 2 are in fact two interpolation results. Their applications to interpolation will be given in section 4 below. To be more "interpolationist", let us introduce the notion of partial retract, a notion we did not know and kindly suggested to us by M. Cwikel. The reader is referred to [1] – [3] for unexplained notions and notations on the interpolation theory.

Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of Banach spaces. We say that (X_0, X_1) is a **partial retract** of (Y_0, Y_1) if for any $x \in X_0 + X_1$ there exist two linear operators $S\colon X_0 + X_1 \to Y_0 + Y_1$ and $T\colon Y_0 + Y_1 \to X_0 + X_1$ such that $TSx = x$ and S (resp. T) is simultaneously bounded from X_i (resp. Y_i) to Y_i (resp. X_i) for $i = 0, 1$. If additionally the norms $\|S\colon X_i \to Y_i\|$ and $\|T\colon Y_i \to X_i\|$

($i = 0, 1$) are bounded by a constant C independent of $x \in X_0 + X_1$, then we say that (X_0, X_1) is a **C-uniformly** (or uniformly, for simplicity) **partial retract** of (Y_0, Y_1).

Remarks: (i) If (X_0, X_1) is a partial retract of (Y_0, Y_1), then interpolation problems on (X_0, X_1) can be brought to those on (Y_0, Y_1). For instance, if this is the case and if (Y_0, Y_1) is a Calderón couple, then (X_0, X_1) is a Calderón couple as well. It is in this way that M. Cwikel [6] proved that a couple of real interpolation spaces between two given spaces is a Calderón couple. In fact, Cwikel showed that such a couple is a uniformly partial retract of a couple of weighted l_p-spaces.

(ii) Theorem 1 shows that (H_1, L_∞) is a uniformly partial retract of (L_1, L_∞). Note that Theorem 1 is somewhat more precise than this latter simple statement, since given $f \in H_1$ we know the value of the associated operator S at f. This precision will be useful in applications. On the other hand, Theorem 2 does not imply that (L_1, BMO) is a partial retract of (L_1, L_∞), but it does imply that (L_p, BMO) is a uniformly partial retract of (L_p, L_∞) for all $1 < p < \infty$ (see section 4).

(iii) Theorems 1 and 2 are motivated by the main result of [14], which says that $(H_1(D), H_\infty(D))$ of analytic functions in the unit disc D is a uniformly partial retract of $(L_1(\mathbf{T}), L_\infty(\mathbf{T}))$ on the unit circle.

Theorems 1 and 2 deal with discrete martingales. In a subsequent paper [15] we will consider martingales in continuous time, in particular, Brownian martingales. The analogues of Theorems 1 and 2 on Brownian martingales in [15] are of special interest since they imply similar results for (H_1, L_∞) and (L_1, BMO) on $\mathbb{R}^n$.

The proofs of Theorems 1 and 2 are presented respectively in the following two sections. They are constructive and based on a refined atomic decomposition already used in [11] and [13]. The construction of the operators T in both theorems is adapted from a similar one in [14] corresponding to the couple $(H_1(D), H_\infty(D))$. Applications to interpolation are given in section 4.

ACKNOWLEDGEMENT: We are grateful to the referee for some useful suggestions.

2. Proof of Theorem 1

From now on, the probability of a measurable subset $A \subset \Omega$ will be denoted by $|A|$. Subsets in the finite partitions generating the $\mathcal{F}_n$'s will be called **cubes**;

for any cube $Q \in \mathcal{F}_n$ $(n \geq 1)$ the unique cube $R \in \mathcal{F}_{n-1}$ containing Q will be denoted by $\tilde{Q}$ and called the double of Q. Note that if Q and R are two cubes such that $Q \cap R \neq \emptyset$, then either $Q \subset R$ or $R \subset Q$.

Now we proceed to prove Theorem 1. The existence of the operators S in both Theorems 1 and 2 follows from the generalized Hahn–Banach theorem, since the maximal function operator and the sharp function operator are positive and sublinear. Alternative simple proofs can be easily found. We have chosen to give a simple proof for S of Theorem 2 in the next section. The reader can modify that proof in order to get S in Theorem 1. Thus we pass directly to the construction of T.

Let $f \in H_1$. It is clear that if $f = f_0$, the multiplication operator by the sign function of f_0 gives the required operator T. Thus we assume $f_0 = 0$. We will decompose f into a sum of atoms with special properties. To this end we need construct a family $\mathcal{C}$ of cubes, which will be defined by induction and by using a standard stopping time argument as in [11].

Let $\Omega_k = \{f^* > 2^k\}$, $k \in \mathbb{Z}$. Then $|\Omega_k| \to 0$ as $k \to +\infty$ and each Ω_k can be written as a disjoint union of maximal cubes contained in it. Here by a maximal cube in a subset A we mean a cube Q such that $Q \subset A$ and its double $\tilde{Q}$ is not contained in A. Note that any maximal cube in Ω_k is contained in a unique maximal cube in Ω_{k-1}.

As the first step of our induction we include in $\mathcal{C}$ all cubes in $\mathcal{F}_0$; then for any $Q_0 \in \mathcal{F}_0$ we set (recalling that M is the constant in (1))

$$k(Q_0) = \inf\left\{k\colon |\Omega_k \cap Q_0| < \frac{1}{2M}|Q_0|\right\}.$$

Writing $\Omega_{k(Q_0)}$ as the disjoint union of its maximal cubes, we let $\mathcal{C}'(Q_0)$ be the family of those maximal cubes of $\Omega_{k(Q_0)}$ contained in Q_0 and $\mathcal{C}(Q_0)$ the family of the maximal cubes in $\{\tilde{Q}\colon Q \in \mathcal{C}'(Q_0)\}$. Then we include in $\mathcal{C}$ all cubes of $\mathcal{C}(Q_0)$ for any $Q_0 \in \mathcal{F}_0$.

As the second step, we take any cube $Q \in \mathcal{C}(Q_0)$ for each fixed $Q_0 \in \mathcal{F}_0$. Then Q is the double of some cube $Q' \in \mathcal{C}'(Q_0)$. Define

$$k(Q) = \inf\left\{k\colon |\Omega_k \cap Q'| < \frac{1}{2M}|Q'|\right\}.$$

As above, let $\mathcal{C}'(Q)$ be the family of the maximal cubes of $\Omega_{k(Q)}$ contained in Q' and $\mathcal{C}(Q)$ the family of the maximal cubes in $\{\tilde{R}\colon R \in \mathcal{C}'(Q)\}$. Then all cubes of $\mathcal{C}(Q)$ are included in $\mathcal{C}$.

Then for any cube already added in $\mathcal{C}$ and not in $\mathcal{F}_0$ we continue the same procedure as in the second step. In this way we finish the induction and thus the construction of the family $\mathcal{C}$ which satisfies the following properties:

(5) Each $Q \in \mathcal{C}$ corresponds to an integer $k(Q)$ such that $Q = \tilde{Q}'$ for some cube Q' (if $Q \in \mathcal{F}_0$, $Q' = Q$ and $\tilde{Q}' = Q'$ by convention) and such that

$$|Q' \cap \Omega_{k(Q)}| < \frac{1}{2M}|Q'| \qquad \text{and} \qquad |Q' \cap \Omega_{k(Q)-1}| \geq \frac{1}{2M}|Q'|.$$

(6) If $Q, R \in \mathcal{C}$ are such that $Q \subset R$ and $Q \neq R$, then $k(Q) > k(R)$.

(7) Each $Q \in \mathcal{C}$ corresponds to a family $\mathcal{C}(Q)$ of disjoint cubes contained in Q such that

$$\sum_{R \in \mathcal{C}(Q)} |R| \leq \frac{1}{2}|Q|.$$

The above properties (5) and (6) are already contained in the construction of $\mathcal{C}$. (7) is established as follows

$$\sum_{R \in \mathcal{C}(Q)} |R| \leq \sum_{R' \in \mathcal{C}'(Q)} |\tilde{R}'| \leq M \sum_{R' \in \mathcal{C}'(Q)} |R'|$$
$$= M|Q' \cap \Omega_{k(Q)}| \leq \frac{1}{2}|Q'| \leq \frac{1}{2}|Q|.$$

For $Q \in \mathcal{C}$ we set

$$A(Q) = Q \setminus \bigcup_{R \in \mathcal{C}(Q)} R.$$

Then $\{A(Q) \colon Q \in \mathcal{C}\}$ is a family of disjoint subsets of Ω with union equal to Ω. Now we can define the atoms that we have promised. For $Q \in \mathcal{C}$ let

$$\alpha_Q = \frac{1}{|A(Q)|} \int_Q f \quad \text{and} \quad a_Q = (f - \alpha_Q)\chi_{A(Q)} + \sum_{R \in \mathcal{C}(Q)} \alpha_R \chi_{A(R)},$$

where χ_e denotes the indicator function of a subset e in Ω. The definition of a_Q is motivated by [13]. Then the family $\{a_Q \colon Q \in \mathcal{C}\}$ enjoys the properties:

(8) $f = \sum_{Q \in \mathcal{C}} a_Q$.

(9) Each a_Q is supported in Q with $\int a_Q = 0$ and $\|a_Q\|_\infty \leq 5 \cdot 2^{k(Q)}$.

(10) $\| \sum_{Q \in \mathcal{C}} 2^{-k(Q)} |a_Q| \|_\infty \leq 10$.

Since $f_0 = 0$, $\alpha_Q = 0$ for any $Q \in \mathcal{F}_0$; that yields (8). It is clear that each a_Q is supported in $A(Q) \cup (\bigcup_{R \in \mathcal{C}(Q)} A(R))$ and so in Q and with mean value equal to 0. If $Q \in \mathcal{C}$ is not in $\mathcal{F}_0$, then $Q \in \mathcal{C}(Q_0)$ for some $Q_0 \in \mathcal{C}$; so $Q = \tilde{Q}'$ for some $Q' \in \mathcal{C}'(Q_0)$. Note that Q' is a maximal cube in $\Omega_{k(Q_0)}$. Hence Q is not contained in $\Omega_{k(Q_0)}$. Then it follows that

$$\left|\frac{1}{|Q|}\int_Q f\right| \le 2^{k(Q_0)}.$$

Now by (7), $|A(Q)| \ge \frac{1}{2}|Q|$. Therefore, $|\alpha_Q| \le 2 \cdot 2^{k(Q_0)}$. On the other hand, if $\omega \in A(Q)$, then $\omega \notin \Omega_{k(Q)}$; so $|f\chi_{A(Q)}| \le 2^{k(Q)}$. Therefore, by $k(Q) > k(Q_0)$, we obtain

$$|a_Q| \le \left(2^{k(Q)} + 2 \cdot 2^{k(Q_0)}\right)\chi_{A(Q)} + \sum_{R \in \mathcal{C}(Q)} 2 \cdot 2^{k(Q)}\chi_{A(R)} \le 5 \cdot 2^{k(Q)}.$$

This gives (9). Finally, (10) follows from (9) and the fact that at most two of the supports of the a_Q's overlap.

In order to define the desired operator T, we need one more notation. For $Q \in \mathcal{C}$ let

$$e(Q) = Q \cap \Omega_{k(Q)-1} \setminus \bigcup_{R \in \mathcal{C}(Q)} (R \cap \Omega_{k(R)-1}).$$

The $e(Q)$'s are disjoint and by (5)

$$\sum_{R \in \mathcal{C}, R \subset Q} |e(R)| = \left|\bigcup_{R \in \mathcal{C}, R \subset Q} e(R)\right| = |Q \cap \Omega_{k(Q)-1}| \ge \frac{1}{2M^2}|Q|. \tag{11}$$

Now we are ready to define T. Let $Q \in \mathcal{C}$. Define

$$T_Q(g) = \frac{1}{|Q|} \sum_{R \in \mathcal{C}, R \subset Q} 2^{-k(R)} \int_{e(R)} g, \quad \forall\, g \in L_1.$$

Note that $f^* > 2^{k(R)-1}$ on $e(R)$; so by (11)

$$T_Q(f^*) \ge \frac{1}{2|Q|} \sum_{R \in \mathcal{C}, R \subset Q} |e(R)| \ge \frac{1}{4M^2}.$$

Let $\lambda_Q = 1/T_Q(f^*)$. Then $0 < \lambda_Q \le 4M^2$. Finally, we define T by

$$T(g) = \sum_{Q \in \mathcal{C}} \lambda_Q T_Q(g) a_Q, \quad \forall\, g \in L_1.$$

We are going to check T satisfies the requirement of Theorem 1. It is clear that $T(f^*) = f$. Now let $g \in L_\infty$. Then by (6)

$$|T_Q(g)| \le \frac{\|g\|_\infty}{|Q|} \sum_{R\in\mathcal{C}, R\subset Q} 2^{-k(R)}|e(R)|$$
$$\le \frac{\|g\|_\infty}{|Q|} 2^{-k(Q)}|Q \cap \Omega_{k(Q)-1}| \le \|g\|_\infty 2^{-k(Q)}.$$

Hence, by (10)

$$\|T(g)\|_\infty \le 4M^2\|g\|_\infty \|\sum_{Q\in\mathcal{C}} 2^{-k(Q)}|a_Q|\|_\infty \le 40M^2\|g\|_\infty,$$

which shows that $\|T\colon L_\infty \to L_\infty\| \le 40M^2$.

For the boundedness of T from L_1 to H_1, we first note by (9) that for each $Q \in \mathcal{C}$, $(5\cdot 2^{k(Q)}|Q|)^{-1}a_Q$ is an atom; so $\|a_Q\|_{H_1} \le 5\cdot 2^{k(Q)}|Q|$. Let $g \in L_1$. Then we have

$$\|T(g)\|_{H_1} \le \sum_{Q\in\mathcal{C}} \lambda_Q |T_Q(g)|\cdot 5\cdot 2^{k(Q)}|Q|$$
$$\le 20M^2 \sum_{Q\in\mathcal{C}} 2^{k(Q)} \cdot \sum_{R\subset Q} 2^{-k(R)} \int_{e(R)} |g|$$
$$= 20M^2 \sum_{R\in\mathcal{C}} 2^{-k(R)} \int_{e(R)} |g| \sum_{Q\supset R} 2^{k(Q)}$$
$$\le 20M^2 \sum_{R\in\mathcal{C}} 2^{-k(R)} \int_{e(R)} |g|\cdot 2^{k(R)+1} \quad \text{(by (6))}$$
$$\le 40M^2\|g\|_1.$$

Therefore, $\|T\colon L_1 \to H_1\| \le 40M^2$, which finishes the proof of Theorem 1.

3. Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1. Now we work with the sharp function $f^\sharp$ instead of the maximal function f^*. First let us give a simple proof for the existence of the operator S.

Let $f \in L_1$ and $\epsilon > 0$. Choose a sequence $\{a_n\}$ of positive random variables such that

$$\sup_{n\ge 0} E(|f - f_n||\mathcal{F}_n) \le (1+\epsilon)\sum_{n\ge 0} a_n E(|f - f_n||\mathcal{F}_n), \quad \|\sum_{n\ge 0} a_n\|_\infty \le 1.$$

Let $\operatorname{sgn} z$ denote the sign of a number z. For any $g \in L_1$ we define

$$S(g) = \operatorname{sgn}(f_0)g_0 + u \sum_{n\geq 0} a_n E\big(\operatorname{sgn}(f - f_n)(g - g_n)\big|\mathcal{F}_n\big)$$

where

$$u = \sup_{n\geq 0} E(|f - f_n||\mathcal{F}_n)\Big(\sum_{n\geq 0} a_n E(|f - f_n||\mathcal{F}_n)\Big)^{-1}.$$

Then $Sf = f^\sharp$ and $|Sg| \leq (1+\epsilon)g^\sharp$ for all $g \in L_1$; so an appropriate limit of S as $\epsilon \to 0$ satisfies the requirement of Theorem 2.

To construct the operator T we will keep all the notations in the previous section, just replacing everywhere the maximal function f^* by the sharp function $f^\sharp$; so, now, for example, the set Ω_k is defined by $\Omega_k = \{f^\sharp > 2^k\}$.

Before starting the construction, let us note the following identity:

$$f^\sharp(\omega) = |f_0(\omega)| + \sup_{\omega\in Q} \frac{1}{|Q|}\int_Q \Big|f - \frac{1}{|Q|}\int_Q f\Big|, \quad \forall\, \omega \in \Omega,$$

where the supremum runs over all cubes containing ω. In the following, it is this formula that we will use.

Let $f \in L_1$. As in the proof of Theorem 1 we can, and do assume $f_0 = 0$. Let $\Omega_k = \{f^\sharp > 2^k\}$ and $\mathcal{C}$ be the family of cubes constructed as in the previous section. Then we still have the properties (5)–(7) for $\mathcal{C}$.

For each $Q \in \mathcal{C}$ let

$$\beta_Q = \frac{1}{|Q|}\int_Q f \quad \text{and} \quad b_Q = (f - \beta_Q)\chi_Q - \sum_{R\in\mathcal{C}(Q)} (f - \beta_R)\chi_R.$$

Now the family $\{b_Q\colon Q \in \mathcal{C}\}$ replaces $\{a_Q\colon Q \in \mathcal{C}\}$ in the previous section. It satisfies:

(12) $f = \sum_{Q\in\mathcal{C}} b_Q$.

(13) $\|b_Q\|_1 \leq 2\cdot 2^{k(Q)}|Q|, \quad \forall\, Q \in \mathcal{C}$.

(12) is evident. For (13) we note that $Q \in \mathcal{C}(Q_0)$ for some $Q_0 \in \mathcal{C}$ and so $Q = \tilde{Q}'$ with $Q' \in \mathcal{C}'(Q_0)$. Then Q contains points not belonging to $\Omega_{k(Q_0)}$. Hence

$$\frac{1}{|Q|}\int_Q |f - \beta_Q| \leq 2^{k(Q_0)}.$$

Therefore, by (6) and (7)

$$\|b_Q\|_1 \leq 2^{k(Q_0)}|Q| + \sum_{R\in\mathcal{C}(Q)} 2^{k(Q)}|R| \leq 2\cdot 2^{k(Q)}|Q|.$$

Let $e(Q)$ and T_Q be defined as in the previous section. Then $T_Q(f^\sharp) \geq 1/(4M^2)$. Finally, we define T by

$$T(g) = \sum_{Q\in\mathcal{C}} \lambda_Q T_Q(g) b_Q\,, \quad \forall\, g \in L_1,$$

where $\lambda_Q = 1/T_Q(f^\sharp)$. Hence $0 < \lambda_Q \leq 4M^2$. Then $T(f^\sharp) = f$ and we check, as before, that

$$\|T(g)\|_1 \leq 16M^2\|g\|_1, \quad \forall g \in L_1,$$

which shows that T is bounded from L_1 to itself.

Let us prove the boundedness of T from L_∞ to BMO. Let $g \in L_\infty$. First note that $E\big(T(g)\big|\mathcal{F}_0\big) = 0$; so

$$\|T(g)\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|}\int_Q \Big|T(g) - \frac{1}{|Q|}\int_Q T(g)\Big|,$$

where the supremum is taken over all cubes. Let Q_0 be any fixed cube. We have

$$\begin{aligned}
&\frac{1}{|Q_0|}\int_{Q_0} \Big|T(g) - \frac{1}{|Q_0|}\int_{Q_0} T(g)\Big| \\
&\quad\leq \sum_{Q\cap Q_0\neq\emptyset} \lambda_Q|T_Q(g)|\frac{1}{|Q_0|}\int_{Q_0}\Big|b_Q - \frac{1}{|Q_0|}\int_{Q_0} b_Q\Big| \\
&\quad= \sum_{Q\subset Q_0} \lambda_Q|T_Q(g)|\frac{1}{|Q_0|}\int_{Q_0}\Big|b_Q - \frac{1}{|Q_0|}\int_{Q_0} b_Q\Big| \\
&\quad\quad+ \sum_{Q\supsetneq Q_0} \lambda_Q|T_Q(g)|\frac{1}{|Q_0|}\int_{Q_0}\Big|b_Q - \frac{1}{|Q_0|}\int_{Q_0} b_Q\Big| \\
&\quad= I + II.
\end{aligned}$$

We are going to estimate I and II separately. For I we observe that for any $Q \in \mathcal{C}$ such that $Q \subset Q_0$ we have $\int_{Q_0} b_Q = 0$ and so, by (13),

$$\frac{1}{|Q_0|}\int_{Q_0}\Big|b_Q - \frac{1}{|Q_0|}\int_{Q_0} b_Q\Big| = \frac{1}{|Q_0|}\|b_Q\|_1 \leq \frac{2|Q|}{|Q_0|}2^{k(Q)}.$$

Therefore, it follows, by (7), that

$$I \leq \frac{8M^2\|g\|_\infty}{|Q_0|}\sum_{Q\subset Q_0}|Q| \leq \frac{8M^2\|g\|_\infty}{|Q_0|}\cdot 2|Q_0| = 16M^2\|g\|_\infty.$$

To estimate II note that for each $Q \in \mathcal{C}$, b_Q is constant on R for any $R \in \mathcal{C}$ such that $R \subset Q$. Then it follows that in the sum II at most one term is not equal to 0. Let Q_1 be the least cube in the family $\{Q \in \mathcal{C}\colon Q \supset Q_0 \text{ and } Q \neq Q_0\}$. Then the only possible non-zero term in II is

$$\lambda_{Q_1}|T_{Q_1}(g)|\frac{1}{|Q_0|}\int_{Q_0}\left|b_{Q_1} - \frac{1}{|Q_0|}\int_{Q_0} b_{Q_1}\right|.$$

Hence

$$II \leq 4M^2\|g\|_\infty 2^{-k(Q_1)}\frac{1}{|Q_0|}\int_{Q_0}\left|b_{Q_1} - \frac{1}{|Q_0|}\int_{Q_0} b_{Q_1}\right|.$$

Note that any R in $\mathcal{C}(Q_1)$ contains points not belonging to $\Omega_{k(Q_1)}$; so if $Q_0 \cap R \neq \emptyset$, then $R \subset Q_0$, and thus Q_0 contains points not belonging to $\Omega_{k(Q_1)}$. On the other hand, if Q_0 does not intersect any R in $\mathcal{C}(Q_1)$, then Q_0 does not intersect $\Omega_{k(Q_1)}$. Therefore we deduce that

$$\begin{aligned}
&\frac{1}{|Q_0|}\int_{Q_0}\left|b_{Q_1} - \frac{1}{|Q_0|}\int_{Q_0} b_{Q_1}\right| \\
&\quad\leq \frac{1}{|Q_0|}\int_{Q_0}\left|f - \frac{1}{|Q_0|}\int_{Q_0} f\right| + \sum_{R\in\mathcal{C}(Q_1), R\subset Q_0}\frac{1}{|Q_0|}\int_R |f - \beta_R| \\
&\quad\leq 2^{k(Q_1)} + \frac{1}{|Q_0|}\cdot 2^{k(Q_1)}\sum_{R\in\mathcal{C}(Q_1),\ R\subset Q_0}|R| \leq 2\cdot 2^{k(Q_1)}.
\end{aligned}$$

Combining the preceding inequalities, we obtain $II \leq 8M^2\|g\|_\infty$. Therefore, for any $g \in L_\infty$ and any cube Q_0 we have established

$$\frac{1}{|Q_0|}\int_{Q_0}|T(g) - \frac{1}{|Q_0|}\int_{Q_0} T(g)| \leq 24M^2\|g\|_\infty.$$

Thus $\|T\colon L_\infty \to \mathrm{BMO}\,\| \leq 24M^2$; so the proof of Theorem 2 is complete.

4. Applications to interpolation

We begin with a definition. Let X be a Banach space of integrable functions on $(\Omega, \mathcal{F}, P)$. We set

$$H(X) = \{f \in L_1\colon f^* \in X\} \quad \text{and} \quad \|f\|_{H(X)} = \|f^*\|_X.$$

$H(X)$ is the Hardy space associated to X of martingales relative to $\{\mathcal{F}_n\}_{n\geq 0}$. By (2) and interpolation we may show that if X is an interpolation space between L_p and L_∞ for some $p > 1$, then $H(X) = X$ with equivalence of norms.

In the sequel C will denote a constant independent of functions in consideration.

COROLLARY 1: *Let F be an interpolation functor. Then*

$$F(H_1, L_\infty) = H(F(L_1, L_\infty)).$$

Proof: Let $f \in F(H_1, L_\infty)$. Then by Theorem 1 we have a linear operator S such that $S(f) = f^*$ and such that $S\colon H_1 \to L_1$ and $S\colon L_\infty \to L_\infty$ are both bounded with proper control of norms as in Theorem 1. Hence $S\colon F(H_1, L_\infty) \to F(L_1, L_\infty)$ is also bounded; so

$$\|f^*\|_{F(L_1,L_\infty)} = \|S(f)\|_{F(L_1,L_\infty)} \leq C\|f\|_{F(H_1,L_\infty)}.$$

Therefore, $F(H_1, L_\infty) \subset H(F(L_1, L_\infty))$. The converse inclusion can be proved similarly by using T in Theorem 1. The detail is omitted. ∎

Remarks: (i) Applying Corollary 1 to the real and complex interpolations we obtain the following known results (cf. [7] and [11] respectively for the real and complex interpolations):

$$(H_1, L_\infty)_{\theta,p} = H_p \quad \text{and} \quad (H_1, L_\infty)_\theta = H_p \quad (0 < \theta < 1,\ 1/p = 1 - \theta).$$

(ii) By the Aronszajn–Gagliardo theorem (cf. [2]) for any interpolation space Y between H_1 and L_∞ there exists an interpolation functor F such that $Y = F(H_1, L_\infty)$; so by Corollary 1, $Y = H(F(L_1, L_\infty))$. This shows that any interpolation space between H_1 and L_∞ is the Hardy space associated to an interpolation space between L_1 and L_∞.

COROLLARY 2: *Let $f \in H_1$ and S, T be the operators given to f by Theorem 1. Then for any exact interpolation space X between L_1 and L_∞, S is bounded from $H(X)$ to X and T is bounded from X to $H(X)$; moreover, the norms of S and T on these spaces are majorized by a constant depending only on M in (1).*

This corollary follows easily from the Aronszajn–Gagliardo theorem and Corollary 1.

With the help of Corollary 2, we can extend Corollary 1 to a more general setting. Recall that $K(t, x; X_0, X_1)$ denotes the K-functional of a couple (X_0, X_1).

COROLLARY 3: *Let X_0 and X_1 be interpolation spaces between L_1 and L_∞. Let F be an interpolation functor. Then*

(i) $F(H(X_0), H(X_1)) = H(F(X_0, X_1))$;

(ii) *for any* $t > 0$ *and any* $f \in H(X_0) + H(X_1)$

$$K(t, f; H(X_0), H(X_1)) \approx K(t, f^*; X_0, X_1) \quad \text{uniformly in } t \text{ and } f.$$

The proofs of this corollary and the following one are similar to that of Corollary 1, and so they are left to the reader.

Let us recall that an interpolation couple (X_0, X_1) of Banach spaces is a Calderón (or Calderón–Mitjagin) couple if $x, y \in X_0 + X_1$ are such that

$$K(t, y; X_0, X_1) \leq K(t, x; X_0, X_1) \quad \text{for all } t > 0,$$

then there exists a bounded linear operator T from X_i to itself for $i = 0, 1$ such that $T(x) = y$. The well-known Calderón theorem [4] says that (L_1, L_∞) is a Calderón couple.

COROLLARY 4: *Let* (X_0, X_1) *be a couple of interpolation spaces of* (L_1, L_∞). *If* (X_0, X_1) *is a Calderón couple, then* $(H(X_0), H(X_1))$ *is a Calderón couple as well.*

In particular, (H_1, L_∞) *is a Calderón couple.*

Remarks: (i) Corollary 4 is the regular martingale version of the corresponding theorems on $(H_1(D), H_\infty(D))$ and $(H_1(\mathbb{R}), L_\infty(\mathbb{R}))$, due to P. Jones [12] and R. Sharpley [13] respectively. Note also that Jones' theorem follows from the main result of [14], already cited in section 1.

(ii) By the K-divisibility theorem of Brudnyi–Krugljak (cf. [2]), the fact that (H_1, L_∞) is a Calderón couple, together with the K-functional characterization of (H_1, L_∞) in Corollary 3, gives a complete description of the interpolation spaces of (H_1, L_∞). In particular, this recovers Corollary 1. Here, without appealing to the Brudnyi–Krugljak theorem we have completely described the interpolation spaces of (H_1, L_∞) directly by Theorem 1.

Let us mention an application of Theorem 1 to the complex interpolation for families of Hardy spaces.

Let D denote the unit disc of the complex plane and $\mathbf{T}$ its boundary with the normalized Lebesgue measure. Let $\{X_\zeta\colon \zeta \in \mathbf{T}\}$ be an interpolation family of Banach spaces in the sense of [5]. Then for any $z \in D$, $\{X_\zeta\}[z]$ denotes one of the interpolation spaces constructed in [5]. It is known that if $\{X_\zeta\colon \zeta \in \mathbf{T}\}$ consists only of two spaces, $\{X_\zeta\}[z]$ becomes the classical complex interpolation space of Calderón. The reader is referred to [5] for more information.

COROLLARY 5: *Let $p(\zeta)$ be a measurable function on $\mathbf{T}$ with $1 \le p(\zeta) \le \infty$, then*

$$\{H_{p(\zeta)}\}[z] = H_{p(z)}\,, \quad \forall\, z \in D,$$

where $1/p(z) = \int_{\mathbf{T}} \big(1/p(\zeta)\big) P_z(\zeta) d\zeta$, P_z being the Poisson kernel of D at z.

Proof: Let $f \in H_{p(z)}$. Then by Theorem 1 we have an operator T such that $Tf^* = f$ and such that $T\colon L_1 \to H_1$ and $T\colon L_\infty \to L_\infty$ are bounded; so $T\colon L_{p(\zeta)} \to H_{p(\zeta)}$ is bounded for all $\zeta \in \mathbf{T}$ and of norm bounded by a constant C. Then by the interpolation theorem of [5], $T\colon \{L_{p(\zeta)}\}[z] \to \{H_{p(\zeta)}\}[z]$ is also bounded and of norm $\le C$. Recall that $\{L_{p(\zeta)}\}[z] = L_{p(z)}$ with equal norms. Therefore

$$\|f\|_{\{H_{p(\zeta)}\}[z]} \le C\|f^*\|_{\{L_{p(\zeta)}\}[z]} = C\|f^*\|_{L_{p(z)}} = C\|f\|_{H_{p(z)}}.$$

Hence, we get the inclusion $H_{p(z)} \subset \{H_{p(\zeta)}\}[z]$. The reverse inclusion is proved similarly. ∎

Now we turn to applications of Theorem 2. Let us first formulate its following consequence.

COROLLARY 6: *(L_p, BMO) is a uniformly partial retract of (L_p, L_∞) for all $1 < p < \infty$.*

Corollary 6 has applications on the couple (L_p, BMO) similar to those of Theorem 1 on (H_1, L_∞) given previously. Here we just mention the following

COROLLARY 7: *Let $1 < p < \infty$. Then (L_p, BMO) is a Calderón couple.*

We end this section by a final remark on (L_1, BMO).

Remark: Although Theorem 2 does not imply that (L_1, BMO) is a partial retract of (L_1, L_∞), it does identify the real intrpolation spaces between L_1 and BMO. For example, by Theorem 2 we can easily get the following result (cf. [10]):

$$(L_1, \mathrm{BMO})_{\theta,p} = L_p \quad \text{with } 1/p = 1-\theta \quad (0<\theta<1).$$

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